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On continuity and compactness of some nonlinear operators in the spaces of functions of bounded variation

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Abstract In this paper, we deal with one of the basic problems of the theory of autonomous superposition operators acting in the spaces of functions of bounded variation, namely the problem concerning their continuity. We basically consider autonomous superposition operators generated by analytic functions or functions of C^1-class. We also investigate the problem of compactness of some classical linear and nonlinear operators acting in the space of functions of bounded variation in the sense of Jordan. We apply our results to the examination of the existence and the topological properties of solutions to nonlinear equations in those spaces.

Keywords Acting condition · Aronszajn-type theorem · Autonomous (nonautonomous) superposition operator · Bernstein polynomials · Compact operator · Hammerstein integral equation · Linear integral operator · Locally bounded mapping · Modulus of continuity · p-variation · Positive solution · R_δ-set · Variation in the sense of Jordan · Volterra–Hammerstein integral equation · ϕ-function · ϕ-variation

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1 Introduction

In the recently published monograph [2], on p. 6 there are stated three basic problems concerning nonautonomous superposition operators acting in the space of functions of bounded variation in the sense of Jordan. The first problem concerns necessary and sufficient conditions which would guarantee that the nonautonomous superposition operator maps the space of functions of bounded variation in the sense of Jordan into itself and is locally bounded. In the paper [7] Bugajewska et al. have given the answer to this problem proving, in particular, the following

Theorem 1 *Suppose that* $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ *is a given function. The following conditions are equivalent:*

1. *the nonautonomous superposition operator* F*, generated by* f*, maps the space* $BV[0,1]$ *into itself and is locally bounded;*
2. *for every* $r>0$*, there exists a constant* $M_r>0$ *such that for every* $k\in\mathbb{N}$*, every finite partition* $0=t_0<\cdots<t_k=1$ *of the interval* $[0,1]$ *and every finite sequence* $u_0,u_1,\ldots,u_k\in[-r,r]$ *with* $\sum_{i=1}^{k}|u_i-u_{i-1}|\le r$*, the following inequalities hold*

$$\sum_{i=1}^{k}\big|f(t_i,u_i)-f(t_{i-1},u_i)\big|\le M_r \quad \textit{and} \quad \sum_{i=1}^{k}\big|f(t_{i-1},u_i)-f(t_{i-1},u_{i-1})\big|\le M_r.$$

(In the above theorem, $BV[0,1]$ denotes the Banach space of all functions $x\colon[0,1]\to\mathbb{R}$ of bounded variation in the sense of Jordan endowed with the norm $\|x\|_{BV}=|x(0)|+\bigvee_0^1 x$; for more details see Sect. 2.)

The third problem mentioned in the monograph [2] concerns the continuity of autonomous and nonautonomous superposition operators described above. In connection with this problem, we would like to draw the reader's attention to the paper by Morse [23]. In Remark 5, we provide more comments on Morse's results contained in that paper.

In the first part of this paper, we deal with the continuity of autonomous superposition operators acting in the space of functions of bounded variation in the sense of Jordan. In particular, we prove that if the generator of an autonomous superposition operator is analytic or of C^1-class, then that operator is continuous. Let us emphasize that our approach to the problem of continuity of autonomous superposition operator is much simpler than that proposed by Morse. It is based on some techniques developed by us in connection with the investigation of mappings of higher order in the spaces of functions of bounded variation in the sense of Jordan as well as on Bernstein polynomials.

In Chapter 7 of the monograph [2], the authors discuss some applications of functions of bounded variation and nonlinear superposition operators acting in such spaces of functions to the theory of nonlinear integral equations. These applications contain and develop the existence and uniqueness results from the paper [8]. Let us also add that the proofs of all those results are based on the Banach contraction principle.

In this paper, we are going to establish that the basic conditions considered in the paper [8] imply the compactness of the Hammerstein integral operator as well as the Volterra–Hammerstein integral operator. As a consequence of this fact, one can use the Schauder-type fixed-point theorems to prove the existence of solutions to nonlinear integral equations under consideration in the classes of functions of bounded variation in the sense of Jordan. In the case of nonlinear Volterra–Hammerstein integral equation, we are also able to describe the topological structure of continuous solution sets to that equation which are of bounded variation in the sense of Jordan. According to our best knowledge, it is the first attempt to establish an Aronszajn-type result for such solutions.

Let us emphasize that the investigation of solutions to nonlinear integral equations in the spaces of functions of bounded variation seems to be interesting for at least a few reasons. First, solutions to many nonlinear equations which describe concrete physical phenomena are functions of bounded variation in the sense of Jordan. We refer the reader to the papers [5] and [18] in which the authors deal with bounded variation solutions to nonlinear Volterra integral equations which describe, in particular, models of behavior of population, where a probability of death depends on age.

Finally, let us mention that functions of bounded variation also possess essential applications, for example, in the geometric measure theory (see, e.g., [1,22]), in image processing, analysis and recovery (see, e.g., [11,12,15,16,27]) or in the theory of Fourier series (see [30]). Let us add that some of those applications are based on the usage of the Mumford–Shah functional.

2 Preliminaries

Notation By $\mathbb{N}$, we denote the set of positive integers. Moreover, throughout the paper by I, we will denote the unit interval [0, 1].

The closed ball in a normed space X with center at x and radius $r \in (0, +\infty)$ will be denoted by $B_X(x, r)$. For simplicity, instead of $B_{\mathbb{R}}(x, r)$, we will simply write $[x - r, x + r]$. As usual, by $L^p(J)$ we will denote the Banach space of all the equivalence classes of real-valued functions defined on a bounded interval $J \subseteq \mathbb{R}$ which are either Lebesgue integrable with pth power, if $p \in [1, +\infty)$, or essentially bounded, if $p = +\infty$, endowed with the norms $\|f\|_{L^p} := \big(\int_J |f(t)|^p dt\big)^{1/p}$ and $\|f\|_{L^\infty} := \operatorname{ess\,sup}_{t\in J} |f(t)|$, respectively. The Lebesgue measure will be denoted by μ. By $B(I)$, we will denote the Banach space of all bounded functions $x : I \to \mathbb{R}$, endowed with the supremum norm $\|x\|_\infty = \sup_{t\in I} |x(t)|$, and by $C(I)$ its closed subspace consisting of all the continuous functions defined on I.

Throughout the paper by '$\int$,' we will denote the Lebesgue integral.

In the next two subsections, we collect basic definitions and facts which will be needed in the sequel.

2.1 Functions of bounded variation

Definition 1 Let $p \in [1, +\infty)$ and let x be a real-valued function defined on I. The number

$$\operatorname{var}_p x = \sup \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})|^p,$$

where the supremum is taken over all finite partitions $0 = t_0 < t_1 < \cdots < t_n = 1$ of I, is called the *p-variation* of the function x over I.

Remark 1 If $p = 1$, then the number $\operatorname{var}_1 x$ is often referred to as the *variation in the sense of Jordan* of the function x and usually is denoted by the symbol $\bigvee_0^1 x$. If $1 \le p \le q$, then $(\operatorname{var}_q x)^{1/q} \le (\operatorname{var}_p x)^{1/p}$ (see for example [13, Remark 2.5] or [21, p. 55]).

Definition 2 A function $\phi : [0, +\infty) \to [0, +\infty)$ is said to be a *ϕ-function*, if it is continuous, unbounded, non-decreasing and such that $\phi(u) = 0$ if and only if $u = 0$.

Definition 3 Let x be a real-valued function defined on I and let ϕ be a given ϕ-function. The number

$$\operatorname{var}_\phi x = \sup \sum_{i=1}^{n} \phi(|x(t_i) - x(t_{i-1})|),$$

where the supremum is taken over all finite partitions $0 = t_0 < t_1 < \cdots < t_n = 1$ of I, is called the *ϕ-variation* (or *variation in the sense of Young*) of the function x over I.

In the sequel, unless stated otherwise, we will assume that all the ϕ-functions are convex.

Remark 2 It is well known that the following vector spaces of functions of bounded variation become Banach spaces, when endowed with the indicated norms:

- $BV_p(I) = \{x : I \to \mathbb{R} : \operatorname{var}_p x < +\infty\}$ with the norm $\|x\|_{BV_p} = |x(0)| + (\operatorname{var}_p x)^{1/p}$;
- $BV_\phi(I) = \{x : I \to \mathbb{R} : x(0) = 0 \text{ and } \operatorname{var}_\phi(\lambda x) < +\infty \text{ for some } \lambda > 0\}$ with the norm $\|x\|_\phi = \inf\{\lambda > 0 : \operatorname{var}_\phi(x/\lambda) \le 1\}$

(see [25, Theorem 3.21]).

Remark 3 Let $(X, \|\cdot\|_X)$ denote $(BV_p(I), \|\cdot\|_{BV_p})$ or $(BV_\phi(I), \|\cdot\|_\phi)$. If $x \in X$, then x is a Lebesgue measurable function (see [24, Theorem 10.7 (a) and Theorem 10.9]), and furthermore, there exists a positive constant $c > 0$ such that $\|x\|_\infty \le c\,\|x\|_X$ for every $x \in X$. For example, if $X = BV_1(I)$, then $c = 1$.

2.2 Bernstein polynomials

Let us recall that the *Bernstein polynomial* of order $n \in \mathbb{N}$ of a function $f \in C(I)$ is defined by the formula

$$B_n(f)(t) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) t^k (1-t)^{n-k} \qquad \text{for } t \in I. \tag{1}$$

If $a > 0$ and $f : [-a, a] \to \mathbb{R}$, we can modify the formula (1) to obtain Bernstein polynomials of the function f on the interval $[-a, a]$, which are given by the formulas

$$B_n^a(f)(t) = \frac{1}{(2a)^n} \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{(2k-n)a}{n}\right) (t+a)^k (a-t)^{n-k} \quad \text{for } t \in [-a, a] \text{ and } n \in \mathbb{N}.$$

In the sequel, we will use the following properties of Bernstein polynomials.

Proposition 1 (cf. [19, Theorem 1.1.1]) *Let $a > 0$ and let $f : [-a, a] \to \mathbb{R}$ be a continuous function. Then the sequence of Bernstein polynomials $(B_n^a(f))_{n\in\mathbb{N}}$ converges uniformly to f on $[-a, a]$.*

Proposition 2 (cf. [19, Section 1.8]) *Let $a > 0$ and let $f : [-a, a] \to \mathbb{R}$ be a continuously differentiable function. Then the sequence of derivatives of Bernstein polynomials $(\frac{d}{dt} B_n^a(f))_{n\in\mathbb{N}}$ converges uniformly to f' on $[-a, a]$.*

Proposition 3 (cf. [19, Section 2.1]) *Let $a > 0$ and $q \in (1, +\infty)$. If $f : [-a, a] \to \mathbb{R}$ is an absolutely continuous function such that $f' \in L^q[-a, a]$, then the sequence of derivatives of Bernstein polynomials $(\frac{d}{dt} B_n^a(f))_{n\in\mathbb{N}}$ converges to f' with respect to the L^q-norm.*

Let us remark that the above properties were established in [19] for Bernstein polynomials of a function defined on the interval I, but they can be easily extended to Bernstein polynomials of functions defined on any interval of the form $[-a, a]$.

For thorough treatment of approximation by Bernstein polynomials, we refer the reader to, for example, [14,19].

3 Continuity of the autonomous superposition operator

Actually, the first result in this section is strongly connected with [9, Theorem 4.1] in which gives the necessary and sufficient condition for an autonomous superposition operator acting in the Banach space $BV_1(I)$ to be a mapping of higher order.

Proposition 4 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a sum of a power series centered at 0 with the radius of convergence $\rho = +\infty$, that is, there exist real numbers $a_0, a_1, \ldots$ such that*

$$f(u) = \sum_{i=0}^{\infty} a_i u^i \qquad \text{for } u \in \mathbb{R}.$$

Then the autonomous superposition operator F, generated by f, which maps the Banach space $BV_1(I)$ into itself, is continuous.

Proof Let us note that the operator F is well defined, since the function f satisfies a local Lipschitz condition (cf. also [3, Theorem 6.13, p. 174]).

For simplicity, let us denote

$$f_n(u) = \sum_{i=0}^{n} a_i u^i, \qquad g_n(u) = \sum_{i=1}^{n} i a_i u^{i-1},$$

where $u \in \mathbb{R}$ and $u^0 = 1$. Furthermore, let $x^0(t) \equiv 1$ and

$$F_n(x) = \sum_{i=0}^{n} a_i x^i \qquad \text{for every } n \in \mathbb{N} \text{ and } x \in BV_1(I).$$

Since $BV_1(I)$ is a Banach algebra under certain norm equivalent to $\|\cdot\|_{BV_1}$ (cf. [3, p. 173] and [21]), the mapping $F_n\colon BV_1(I) \to BV_1(I)$ is continuous. Therefore, in order to show the continuity of F, it suffices to show that the sequence of mappings $(F_n)_{n\in\mathbb{N}}$ converges to F uniformly on bounded sets. Since the function $u \mapsto (f_n - f)(u)$ satisfies the Lipschitz condition on every interval $[-a, a]$ with the constant $L_n(a) = \sup_{u\in[-a,a]} |f'(u) - g_n(u)|$, we obtain

$$\bigvee_0^1 [F_n(x) - F(x)] \le L_n(b) \bigvee_0^1 x,$$

where $b = \|x\|_{BV_1}$. Hence

$$\begin{aligned}\|F_n(x) - F(x)\|_{BV_1} &= |f_n(x(0)) - f(x(0))| + \bigvee_0^1 [F_n(x) - F(x)] \\ &\le |f_n(x(0)) - f(x(0))| + L_n(a) \bigvee_0^1 x,\end{aligned}$$

which shows that $F_n(x) \to F(x)$ as $n \to +\infty$ uniformly on $B_{BV_1}(0, a)$, where a is an arbitrary but fixed positive real number. This ends the proof. □

The assumption of the analyticity of the generator of the autonomous superposition operator which appears in Proposition 4 can be replaced by a weaker assumption that this generator is of C^1-class.

Theorem 2 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Then the autonomous superposition operator $F\colon BV_1(I) \to BV_1(I)$, generated by the function f, is continuous.*

Proof First, let us observe that the autonomous superposition operator F is well defined. In order to show its continuity, similarly to the proof of Proposition 4, we will approximate F by an almost uniformly convergent sequence of continuous mappings on $BV_1(I)$.

For a given $a > 0$, let φ_a denote the restriction of the function f to the interval $[-a, a]$, that is, $\varphi_a(u) = f|_{[-a,a]}(u)$ for $u \in [-a, a]$. Moreover, let $F_n\colon B_{BV_1}(0, a) \to BV_1(I)$ be the autonomous superposition operator generated by the n-th order Bernstein polynomial $B_n^a(\varphi_a)$ of the function φ_a. Since $BV_1(I)$ is a Banach algebra under certain norm equivalent to $\|\cdot\|_{BV_1}$, the operators F_n are clearly continuous.

Now, we are going to show that the sequence $(F_n)_{n\in\mathbb{N}}$ converges uniformly to F on $B_{BV_1}(0, a)$. Let us note that the function $u \mapsto [f - B_n^a(\varphi_a)](u)$ satisfies the Lipschitz condition on the interval $[-a, a]$ with the constant $L_n(a) = \sup_{u\in[-a,a]}\big|f'(u) - \frac{\mathrm{d}}{\mathrm{du}}B_n^a(\varphi_a)(u)\big|$, and hence, we have

$$\bigvee_0^1 \big[F(x) - F_n(x)\big] \le L_n(a) \bigvee_0^1 x \qquad \text{for } x \in B_{BV_1}(0, a).$$

Therefore, by Proposition 1 and Proposition 2, for $x \in B_{BV_1}(0, a)$, we get

$$\begin{aligned}\|F(x) - F_n(x)\|_{BV_1} &\le \big|f(x(0)) - B_n^a(\varphi_a)(x(0))\big| + L_n(a) \bigvee_0^1 x \\ &\le \big|\varphi_a(x(0)) - B_n^a(\varphi_a)(x(0))\big| + aL_n(a) \to 0,\end{aligned}$$

as $n \to +\infty$, which ends the proof. □

It turns out that a result similar to Theorem 2 may also be established for a wider class of generators $f\colon \mathbb{R} \to \mathbb{R}$, under the additional cost of weakening the topology of the target space.

Theorem 3 *Suppose that a function $f\colon \mathbb{R} \to \mathbb{R}$ is absolutely continuous on each compact subinterval of $\mathbb{R}$. Moreover, assume that there exists a number $q \in (1, +\infty)$ such that $f' \in L^q[-a, a]$ for every $a > 0$. Then the autonomous superposition operator $F\colon BV_1(I) \to BV_p(I)$ is continuous, where p is the conjugate number to q, that is, $p^{-1} + q^{-1} = 1$.*

Proof Fix $a > 0$. First, let us observe that for arbitrary real numbers $u, w \in [-a, a]$ such that $u < w$, by the Hölder inequality, we get

$$\begin{aligned}|f(u) - f(w)|^p &\le \left(\int_u^w |f'(\tau)|\mathrm{d}\tau\right)^p \le |u - w| \cdot \left(\int_u^w |f'(\tau)|^q \mathrm{d}\tau\right)^{\frac{p}{q}} \\ &\le \|f'\|^p_{L^q[-a,a]}\, |u - w|. \end{aligned} \tag{2}$$

Therefore

$$\operatorname{var}_p F(x) \le \|f'\|^p_{L^q[-a,a]} \cdot \bigvee_0^1 x \qquad \text{for every } x \in B_{BV_1}(0, a),$$

which shows that the superposition operator F, generated by the function f, is well defined.

Now, we proceed as in the proof of Theorem 2. Fix $a > 0$ and denote by φ_a the restriction of f to the interval $[-a, a]$. Since, the Banach space $BV_1(I)$ is continuously embedded into $BV_p(I)$ (see Remark 1), we infer that the superposition operators $F_n\colon B_{BV_1}(0,a) \to BV_p(I)$, generated by the Bernstein polynomials of the function φ_a, are continuous.

As before, to end the proof, it suffices to show that the sequence $(F_n)_{n\in\mathbb{N}}$ converges uniformly on $B_{BV_1}(0,a)$ to F. Let us observe that for every $n \in \mathbb{N}$, the function $\psi_n^a = \varphi_a - B_n^a(\varphi_a)$ is absolutely continuous on $[-a,a]$ and its derivative is q-th power Lebesgue integrable. Hence

$$|\psi_n^a(u) - \psi_n^a(w)|^p \leq \left\| \tfrac{\mathrm{d}}{\mathrm{d}u}\psi_n^a \right\|^p_{L^q[-a,a]} \cdot |u-w| \qquad \text{for} \quad u, w \in [-a,a]$$

(cf. the formula (2)). This in turn implies that for every $x \in B_{BV_1}(0,a)$, we get

$$\|F(x) - F_n(x)\|_{BV_p} \leq \left| f(x(0)) - B_n^a(\varphi_a)(x(0)) \right| + a^{\frac{1}{p}} \cdot \left\| \tfrac{\mathrm{d}}{\mathrm{d}u}\psi_n^a \right\|_{L^q[-a,a]}.$$

By Proposition 3, we infer that $\left\| \frac{\mathrm{d}}{\mathrm{d}u}\psi_n^a \right\|_{L^q[-a,a]} \to 0$ as $n \to +\infty$, which shows that $F_n \to F$ uniformly on $B_{BV_1}(0,a)$ and ends the proof. □

Since the derivative of a Lipschitz continuous function is essentially bounded (cf. [20, Theorem 7.1.5, p. 150]), from Theorem 3 we obtain the following

Corollary 1 *If $f\colon \mathbb{R} \to \mathbb{R}$ is a function which satisfies a local Lipschitz condition, then the autonomous superposition operator $F\colon BV_1(I) \to BV_1(I)$, generated by f, is BV_1–BV_p continuous for every $p > 1$.*

Remark 4 Let us observe that in Theorems 2 and 3, the Bernstein polynomials may be replaced by any other approximation scheme built on the polynomials approximating the given function as well as its derivative.

Remark 5 It is worth noting that the issue concerning the continuity of a superposition operator in the space $BV_1(I)$ was also addressed by Morse, who in [23] proved that, if a generator f can be decomposed into functions of certain regularity, then the nonautonomous superposition operator, corresponding to the function f, acts in the space $BV_1(I)$ and is continuous (see [23, Theorem 7.1]). It can be easily verified that every locally Lipschitzian function exhibits the above-mentioned decomposition (just take $U(y_1, y_2) = y_2$, $A(t) = t$ and $B(x) = f(x)$), and therefore, Morse's result provides the complete answer to the question concerning the continuity of an autonomous superposition operator in the space $BV_1(I)$ raised by the authors of the monograph [2]. Unfortunately, the problem of the continuity of superposition operators in the nonautonomous case still seems far from being fully solved, since the conditions imposed on the generator f in [23, Theorem 7.1] imply its continuity, whereas it is easy to see that there are nonautonomous superposition operators generated by discontinuous functions which map the space $BV_1(I)$ into itself continuously; it is enough, for example, to consider the function $f\colon I \times \mathbb{R} \to \mathbb{R}$ given by the formula

$$f(t,x) = \begin{cases} x, & \text{if } (t,x) \in \{0\} \times \mathbb{R}, \\ 0, & \text{if } (t,x) \in (0,1] \times \mathbb{R}. \end{cases}$$

Finally, it should also be emphasized that the proof of Morse's theorem is highly non-trivial and, although formally it is only one page long, it is based on several preceding results (which, by the way, are interesting by themselves) and therefore may be considered to be nearly 30 pages long. On the other hand, the proofs of our results, the assumptions of which are sufficient for many applications, are quite short and simple.

Now, we will pass to the more general setting. We will start with recalling some notions connected with moduli of continuity.

Definition 4 (see [14, p. 41]) A function $\omega\colon [0,+\infty) \to [0,+\infty)$ is called a *modulus of continuity*, if it is non-decreasing, subadditive, continuous and $\omega(0) = 0$.

Definition 5 (cf. [4, p. 406]) A modulus of continuity ω is said to be a *modulus of continuity of a function* $f\colon [-a,a] \to \mathbb{R}$, if $|f(t) - f(s)| \leq \omega(\delta)$ for all points t, s in $[-a,a]$ such that $|t - s| \leq \delta$.

Remark 6 Every continuous function $f\colon [-a,a] \to \mathbb{R}$ admits a modulus of continuity, which is given by the following formula

$$\omega_f(\delta) = \sup\bigl\{|f(t) - f(s)| : t, s \in [-a,a] \text{ and } |t - s| \leq \delta\bigr\}, \qquad \delta \geq 0 \tag{3}$$

(cf. [14, p. 40]). The modulus of continuity defined by (3) is often referred to as the *optimal modulus of continuity of the function* f.

In the sequel, we will need the following technical lemma, whose proof we omit, since it is similar to the proof of [14, Lemma 6.1, p. 43].

Lemma 1 *Each continuous function* $f\colon [-a,a] \to \mathbb{R}$ *admits a modulus of continuity* ω^* *which is strictly increasing, unbounded, concave and such that* $\omega_f(\delta) \leq \omega^*(\delta)$ *for* $\delta \geq 0$.

Before we proceed further, we recall the following property of modulus of continuity of Bernstein polynomials.

Proposition 5 (cf. [17]) *If* $f\colon [-a,a] \to \mathbb{R}$ *is a continuous function with the optimal modulus of continuity* ω_f, *then* $\omega_{B_n^a(f)}(t) \leq 4\omega_f(t)$ *for* $t \geq 0$.

Lemma 2 *Let* $\omega\colon [0,+\infty) \to [0,+\infty)$ *be a concave, unbounded and strictly increasing modulus of continuity of a continuous function* $f\colon [-a,a] \to \mathbb{R}$. *Then the autonomous superposition operator* F, *generated by the function* f, *maps the closed ball* $B_{BV_1}(0,a)$ *into the subset of the space* $B(I)$ *consisting of functions of finite* ω^{-1}*-variation.*

Proof For an arbitrary finite partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of the interval I, we have

$$\begin{aligned}\sum_{i=1}^{n} \omega^{-1}\bigl(|f(x(t_i)) - f(x(t_{i-1}))|\bigr) &\leq \sum_{i=1}^{n} \omega^{-1}\bigl(\omega(|x(t_i) - x(t_{i-1})|\bigr) = \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| \\ &\leq \bigvee_0^1 x < +\infty,\end{aligned}$$

so $\mathrm{var}_{\omega^{-1}} F(x) < +\infty$. □

At the end of this section, we are going to prove another result concerning ϕ-variation in the proof of which we use a similar idea as in the proofs of Theorems 2 and 3.

Theorem 4 *Let* $\omega\colon [0,+\infty) \to [0,+\infty)$ *be a concave, unbounded and strictly increasing modulus of continuity of the continuous function* $f\colon [-a,a] \to \mathbb{R}$, *and let us denote by* ϕ *the function* $\phi\colon [0,+\infty) \to [0,+\infty)$ *given by the formula* $\phi(s) = \bigl[\omega^{-1}(\frac{1}{5}s)\bigr]^p$, *where* $p > 1$. *Moreover, by* F *and* F_n, *let us denote the autonomous superposition operators, generated by* f *and* $B_n^a(f)$, *respectively, which map the closed ball* $B_{BV_1}(0,a)$ *into the subset of the space* $B(I)$ *consisting of functions of finite* ϕ*-variation. Then* $\mathrm{var}_\phi[F(x) - F_n(x)] \to 0$ *uniformly on* $B_{BV_1}(0,a)$.

Proof First, we shall show that the superposition operators are well defined, that is, for every $x \in B_{BV_1}(0,a)$ the functions $F(x)$, $F_n(x)$ are of finite ϕ-variation. Indeed, if $0 = t_0 < \cdots < t_m = 1$ is an arbitrary finite partition of the interval I, then

$$\begin{aligned}
&\sum_{i=1}^{m} \phi\big(|f(x(t_i)) - f(x(t_{i-1}))|\big) \\
&\quad = \sum_{i=1}^{m} \big[\omega^{-1}\big(\tfrac{1}{5}\big|f(x(t_i)) - f(x(t_{i-1}))\big|\big)\big]^p \\
&\quad \le \sum_{i=1}^{m} \big[\omega^{-1}\big(\big|f(x(t_i)) - f(x(t_{i-1}))\big|\big)\big]^p \le \sum_{i=1}^{m} \big[\omega^{-1}\big(\omega\big(|x(t_i) - x(t_{i-1})|\big)\big)\big]^p \\
&\quad = \sum_{i=1}^{m} |x(t_i) - x(t_{i-1})|^p \le \bigg(\bigvee_0^1 x\bigg)^p
\end{aligned}$$

(cf. Remark 1). This shows that $F(x)$ is of finite ϕ-variation. To prove that $\operatorname{var}_\phi F_n(x) < +\infty$, it suffices to apply Proposition 5 along with the fact that $\omega_f(t) \le \omega(t)$ for $t \ge 0$ and to follow the above reasoning.

Now, we are going to show that for a given $\varepsilon > 0$ and for all but finitely many $n \in \mathbb{N}$, we have

$$\operatorname{var}_\phi[F(x) - F_n(x)] < \varepsilon, \qquad \text{whenever} \qquad x \in B_{BV_1}(0,a).$$

Fix $\varepsilon > 0$ and choose $\delta > 0$ to be such that $\delta^{p-1} < \varepsilon/a$. Moreover, for an arbitrary finite partition $0 = t_0 < \cdots < t_m = 1$ of the interval I, let us introduce the following sets

$$N_1 = \{i \in \mathbb{N} : |x(t_i) - x(t_{i-1})| \le \delta\} \qquad \text{and} \qquad N_2 = \{i \in \mathbb{N} : |x(t_i) - x(t_{i-1})| > \delta\}.$$

If $i \in N_1$, then

$$\begin{aligned}
&\big|f(x(t_i)) - B_n^a(f)(x(t_i)) - f(x(t_{i-1})) + B_n^a(f)(x(t_{i-1}))\big| \\
&\quad \le \big|f(x(t_i)) - f(x(t_{i-1}))\big| + \big|B_n^a(f)(x(t_i)) - B_n^a(f)(x(t_{i-1}))\big| \\
&\quad \le 5\omega_f\big(|x(t_i) - x(t_{i-1})|\big) \le 5\omega\big(|x(t_i) - x(t_{i-1})|\big),
\end{aligned}$$

and thus

$$\begin{aligned}
\phi\big(\big|[F(x) - F_n(x)](t_i) - [F(x) - F_n(x)](t_{i-1})\big|\big) &\le \phi\big(5\omega\big(|x(t_i) - x(t_{i-1})|\big)\big) \\
&= |x(t_i) - x(t_{i-1})|^p.
\end{aligned}$$

Therefore, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\sum_{i \in N_1} \phi\big(\big|[F(x) - F_n(x)](t_i) - [F(x) - F_n(x)](t_{i-1})\big|\big) \le \sum_{i \in N_1} |x(t_i) - x(t_{i-1})|^p \\
&\quad = \sum_{i \in N_1} |x(t_i) - x(t_{i-1})|^{p-1} |x(t_i) - x(t_{i-1})| \le \delta^{p-1} \sum_{i \in N_1} |x(t_i) - x(t_{i-1})| \\
&\quad \le \varepsilon/a \cdot \bigvee_0^1 x \le \varepsilon.
\end{aligned}$$

In order to estimate the sum over N_2, let us choose a number $\eta > 0$ such that $\phi(2\eta) \le \delta\varepsilon/a$. Furthermore, assume that the inequality $\sup_{s \in [-a,a]} |f(s) - B_n^a(f)(s)| \le \eta$ holds for all

$n \geq n_0$, where $n_0 \in \mathbb{N}$ is a certain positive integer (cf. Proposition 1). Then, for $i \in N_2$ and $n \geq n_0$, we have

$$\begin{aligned}
&\phi\left(\left|[F(x) - F_n(x)](t_i) - [F(x) - F_n(x)](t_{i-1})\right|\right) \\
&\quad \leq \phi\left(\left|f(x(t_i)) - B_n^a(f)(x(t_i))\right| + \left|f(x(t_{i-1})) - B_n^a(f)(x(t_{i-1}))\right|\right) \\
&\quad \leq \phi(2\eta) \leq \frac{\delta\varepsilon}{a} \leq \frac{\varepsilon}{a}|x(t_i) - x(t_{i-1})|.
\end{aligned}$$

Hence

$$\sum_{i \in N_2} \phi\left(\left|[F(x) - F_n(x)](t_i) - [F(x) - F_n(x)](t_{i-1})\right|\right) \leq \frac{\varepsilon}{a} \sum_{i \in N_2} |x(t_i) - x(t_{i-1})| \leq \varepsilon.$$

Hence, for all $n \geq n_0$, we obtain

$$\sum_{i=1}^{m} \phi\left(\left|[F(x) - F_n(x)](t_i) - [F(x) - F_n(x)](t_{i-1})\right|\right) \leq 2\varepsilon,$$

which proves that $\mathrm{var}_\phi[F(x) - F_n(x)] \to 0$ uniformly on $B_{BV_1}(0, a)$. □

4 Compactness results and applications

4.1 Hammerstein integral equation

In this subsection, we will be interested in the problem of the existence of solutions of bounded variation in the sense of Jordan to the following nonlinear Hammerstein integral equation

$$x(t) = \lambda \int_0^1 k(t, s) f(x(s)) \mathrm{d}s, \qquad t \in I, \tag{4}$$

where $\lambda \in \mathbb{R}$. Let us make the following assumptions:

1° $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that:

(a) f is absolutely continuous on each compact subinterval of $\mathbb{R}$;
(b) there exists a number $q \in (1, +\infty)$ such that $f' \in L^q[-a, a]$ for every $a > 0$;
(c) f is sub-linear, that is, $\lim_{|u| \to +\infty} |f(u)|/|u| = 0$;

2° the kernel $k : I \times I \to \mathbb{R}$ is such that:

(a) for every $t \in I$ the function $s \mapsto k(t, s)$ is Lebesgue measurable;
(b) the function $s \mapsto k(0, s)$ is Lebesgue integrable;
(c) $\bigvee_0^1 k(\cdot, s) \leq m(s)$ for a.e. $s \in I$, where $m : I \to [0, +\infty)$ is a Lebesgue integrable function.

Remark 7 Let us recall that the condition of type 2° was introduced in the paper [8]. Moreover, let us note that if the kernel k satisfies the assumptions 2° (a)–(c), then for every $t \in I$, the function $s \mapsto k(t, s)$ is Lebesgue integrable on I. Indeed, given any $t \in I$, we have

$$|k(t, s)| \leq |k(0, s) - k(t, s)| + |k(0, s)| \leq \bigvee_0^1 k(\cdot, s) + |k(0, s)| \leq m(s) + |k(0, s)|$$

for a.e. $s \in I$,

which confirms our claim.

Proposition 6 *Let $p, r \in [1, +\infty)$ and suppose that the kernel $k\colon I \times I \to \mathbb{R}$ satisfies the assumptions 2° (a)–(c). Then the linear integral operator $K\colon BV_p(I) \to BV_r(I)$, defined by the formula*

$$Kx(t) = \int_0^1 k(t, s)x(s)ds, \qquad t \in I,\ x \in BV_p(I), \tag{5}$$

is compact.

Proof For every $x \in BV_p(I)$, in view of the assumptions, Remarks 3 and 7, the integral

$$\int_0^1 k(t, s)x(s)\mathrm{d}s$$

exists and is finite for every $t \in I$. Moreover, if $0 = t_0 < t_1 < \cdots < t_n = 1$ is an arbitrary finite partition of the interval I, then

$$\begin{aligned}\sum_{i=1}^{n} \bigl|Kx(t_i) - Kx(t_{i-1})\bigr| &\le \int_0^1 \sum_{i=1}^{n} |k(t_i, s) - k(t_{i-1}, s)||x(s)|\mathrm{d}s \le \int_0^1 m(s)\, \|x\|_\infty\, \mathrm{d}s \\ &\le m_1 c_p\, \|x\|_{BV_p},\end{aligned}$$

where $m_1 = \int_0^1 m(s)\mathrm{d}s$ and the number c_p is such that $\|x\|_\infty \le c_p\, \|x\|_{BV_p}$ for $x \in BV_p(I)$ (cf. Remark 3). Hence

$$(\mathrm{var}_r\, Kx)^{1/r} \le \bigvee_0^1 Kx \le m_1 c_p\, \|x\|_{BV_p},$$

and therefore $\|Kx\|_{BV_r} \le c_p\bigl(\|k(0,\cdot)\|_{L^1} + m_1\bigr)\, \|x\|_{BV_p}$. This proves that the operator K is well defined and continuous.

Now, we will show that K is compact. If $(x_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of elements of $B_{BV_p}(0, 1)$, then, in view of Helly's selection theorem (see [13, Theorem 6.1;26, Theorem 2.4]), there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, pointwise convergent to some $x \in B_{BV_p}(0, 1)$. Let $y_k = x_{n_k} - x$ for $k \in \mathbb{N}$. We will establish that $Ky_k \to 0$ with respect to the BV_r-norm. Given $\varepsilon > 0$ let $k_0 \in \mathbb{N}$ be such that

$$\int_0^1 |k(0, s)||y_k(s)|\mathrm{d}s \le \frac{1}{2}\varepsilon \qquad \text{and} \qquad \int_0^1 m(s)|y_k(s)|\mathrm{d}s \le \frac{1}{2}\varepsilon \qquad \text{for} \quad k \ge k_0$$

(note that such k_0 exists, since

$$\int_0^1 |k(0, s)||y_k(s)|\mathrm{d}s \to 0 \qquad \text{and} \qquad \int_0^1 m(s)|y_k(s)|\mathrm{d}s \to 0 \qquad \text{as} \quad k \to +\infty$$

by the dominated convergence theorem). If $0 = t_0 < t_1 < \cdots < t_m = 1$ is an arbitrary finite partition of the interval I, then for $k \ge k_0$, we have

$$\sum_{i=1}^{m} \bigl|Ky_k(t_i) - Ky_k(t_{i-1})\bigr| \le \int_0^1 \sum_{i=1}^{m} |k(t_i, s) - k(t_{i-1}, s)||y_k(s)|\mathrm{d}s \le \int_0^1 m(s)|y_k(s)|\mathrm{d}s \le \frac{1}{2}\varepsilon,$$

which implies that

$$\bigvee_0^1 Ky_k \le \frac{1}{2}\varepsilon \qquad \text{for} \quad k \ge k_0.$$

Thus

$$\|Ky_k\|_{BV_r} \leq \left|Ky_k(0)\right| + \bigvee_0^1 Ky_k \leq \int_0^1 |k(0,s)||y_k(s)|\mathrm{d}s + \bigvee_0^1 Ky_k \leq \varepsilon \qquad \text{for } k \geq k_0,$$

which ends the proof. □

Now, we apply the above compactness result as well as the continuity result (Theorem 3) to prove the following existence theorem for Eq. (4).

Theorem 5 *Let the kernel $k\colon I \times I \to \mathbb{R}$ and the nonlinearity $f\colon \mathbb{R} \to \mathbb{R}$ satisfy the assumptions 2° (a)–(c) and 1° (a)–(c), respectively. Then for every $\lambda \in \mathbb{R}$, there exists a BV_1-solution to Eq. (4).*

Proof If $\lambda = 0$, then the claim is obvious. Let us note that without the loss of generality, we may assume that $\lambda = 1$, and let us consider the operator $G = K \circ F\colon BV_1(I) \to BV_1(I)$, where the integral operator $K\colon BV_p(I) \to BV_1(I)$, for $p^{-1} + q^{-1} = 1$, is given by (5), while the autonomous superposition operator $F\colon BV_1(I) \to BV_p(I)$ is generated by the function f. In view of Proposition 6 and Theorem 3, the operator G is continuous. Let us note that it is also completely continuous since F maps bounded sets of $BV_1(I)$ into bounded sets of $BV_p(I)$, which is a direct consequence of the inequality

$$\operatorname{var}_p F(x) \leq \left\| f' \right\|^p_{L^q[-a,a]} \cdot \bigvee_0^1 x \qquad \text{for every} \quad x \in B_{BV_1}(0,a)$$

(cf. the proof of Theorem 3).

Therefore, it is enough to find a closed ball $B_{BV_1}(0,a) \subseteq BV_1(I)$ invariant under the completely continuous mapping G. First, let us observe that for any ball $B_{BV_1}(0,a)$ and $x \in B_{BV_1}(0,a)$ there is $\|x\|_\infty \leq a$ and

$$\|F(x)\|_\infty \leq \sup_{s\in[-a,a]} |f(s)|.$$

By the assumption 1° (c), there exists such $R > 0$ that

$$\sup_{s\in[-R,R]} |f(s)| \cdot \left(\int_0^1 |k(0,s)|\mathrm{d}s + \int_0^1 m(s)\mathrm{d}s \right) \leq R.$$

Otherwise, a sequence $(u_n)_{n\in\mathbb{N}}$ of real numbers would exist for which we would have

$$|f(u_n)| \cdot \left(\int_0^1 |k(0,s)|\mathrm{d}s + \int_0^1 m(s)\mathrm{d}s \right) > n \quad \text{and} \quad |u_n| \leq n.$$

The sequence $(u_n)_{n\in\mathbb{N}}$ may not be bounded, so there has to exist an appropriate subsequence $(w_n)_{n\in\mathbb{N}}$, such that $|w_n| \to +\infty$ and

$$\frac{|f(w_n)|}{|w_n|} \geq \left(\int_0^1 |k(0,s)|\mathrm{d}s + \int_0^1 m(s)\mathrm{d}s \right)^{-1},$$

which contradicts the assumption 1° (c).

Therefore

$$\|G(x)\|_{BV_1} = |G(x)(0)| + \bigvee_0^1 G(x) \leq \int_0^1 |k(0,s)||f(x(s))|\mathrm{d}s + \int_0^1 m(s)|f(x(s))|\mathrm{d}s \leq R$$

for $x \in B_{BV_1}(0, R)$, which implies that the ball $B_{BV_1}(0, R)$ is invariant under the mapping G. This—by the Schauder fixed-point theorem—implies that there exists a fixed point of G and completes the proof. □

Using the techniques developed in the paper [10], we are able to establish the following result concerning the existence of positive continuous solutions to the nonautonomous version of Eq. (4) in the class of functions of bounded variation in the sense of Jordan which satisfy a certain additional condition.

Theorem 6 *Let $J \subseteq I$ be a closed set of positive Lebesgue measure. Moreover, let the functions $k\colon I \times I \to [0, +\infty)$ and $f : I \times [0, r] \to [0, +\infty)$, where $r > 0$, be continuous. If the kernel k satisfies the assumption $2°$ (c) and if there exist positive numbers δ_1, δ_2, M such that:*

(i) $\int_J k(t, s)dt \geq \delta_1$ *for each* $s \in J$;
(ii) $\int_J k(t, s)dt \geq \delta_2 k(u, s)$ *for each* $(u, s) \in I \times I$;
(iii) $f(t, u) > 0$ *if* $M\mu(J)^{-1/p} \leq u \leq r$ *and* $t \in J$, *where* $p > 1$;
(iv) $0 < M \leq r\delta_2\mu(J)^{-1/q}$, *where* $q > 1$ *is such that* $p^{-1} + q^{-1} = 1$,

then there exits $\lambda > 0$ such that the equation

$$x(t) = \lambda \int_0^1 k(t, s) f(s, x(s))ds, \qquad t \in I,$$

has a positive[1] *and continuous BV_1-solution x such that*

$$\left(\int_J [x(t)]^p dt\right)^{1/p} = M.$$

Proof The assertion follows easily from [10, Theorem 12]. □

Example 1 It is easy to check that the kernel $k\colon I \times I \to [0, +\infty)$ given by $k(t, s) = |t - s|$ satisfies the assumption $2°$ (c) as well as the assumptions (i) and (ii) of Theorem 6 with $J = I = [0, 1]$.

4.2 Volterra–Hammerstein integral equation

Let us consider the following nonlinear Volterra–Hammerstein integral equation

$$x(t) = g(t) + \int_0^t k(t, s) f(s, x(s))\mathrm{d}s, \qquad t \in I, \tag{6}$$

where

$3°$ $g \in C(I) \cap BV_1(I)$;
$4°$ $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions, that is,

(a) for every $u \in \mathbb{R}$ the function $t \mapsto f(t, u)$ is Lebesgue measurable;
(b) for a.e. $t \in I$ the function $u \mapsto f(t, u)$ is continuous;
(c) $|f(t, u)| \leq m_1(t)$ for $(t, u) \in I \times \mathbb{R}$ with $m_1 \in L^p(I)$, where $p \in (1, +\infty]$;

$5°$ the kernel $k\colon \Delta \to \mathbb{R}$, where $\Delta := \{(t, s) \in I \times I : 0 \leq s \leq t \leq 1\}$, is such that:

(a) for every $t \in I$ the function $s \mapsto k(t, s)$ is Lebesgue measurable on $[0, t]$;

[1] Let us recall that by a positive solution, we understand a solution which takes only nonnegative values.

(b) $|k(s,s)|+\bigvee_s^1 k(\cdot,s) \le m_2(s)$ for a.e. $s \in I$ with $m_2 \in L^q(I)$, where $q^{-1}+p^{-1}=1$;
(c) for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_0^t |k(\tau,s)-k(t,s)|m_1(s)\mathrm{ds} \le \varepsilon,$$

for all $(\tau,t) \in \Delta$ such that $0 \le \tau - t \le \delta$.

Remark 8 Let us note that if the kernel k satisfies the assumptions 5° (a) and (b), then for every $t \in I$ the function $s \mapsto k(t,s)$ belongs to $L^q[0,t]$. Indeed, given any $t \in I$, we have

$$|k(t,s)| \le |k(s,s)-k(t,s)|+|k(s,s)| \le \bigvee_s^t k(\cdot,s)+|k(s,s)| \le m_2(s) \quad \text{for a.e. } s \in [0,t],$$

which confirms our claim.

Lemma 3 *Let $p \in [1,+\infty]$. If the kernel k satisfies the assumptions 5° (a) and (b), then the linear Volterra integral operator K defined by*

$$Kx(t) = \int_0^t k(t,s)x(s)ds, \qquad t \in I, \tag{7}$$

maps the space $L^p(I)$ into $BV_1(I)$ and is continuous.

Proof Let $x \in L^p(I)$. First, let us observe that in view of Remark 8, the integral

$$\int_0^t k(t,s)x(s)\mathrm{ds}$$

exists and is finite for every $t \in I$, and thus the definition of the operator K does make sense.

If $0 = t_0 < \cdots < t_n = 1$ is an arbitrary finite partition of the interval I, then

$$\begin{aligned}\sum_{i=1}^n |Kx(t_i)-Kx(t_{i-1})| &= \sum_{i=1}^n \left| \int_0^{t_i} k(t_i,s)x(s)\mathrm{ds} - \int_0^{t_{i-1}} k(t_{i-1},s)x(s)\mathrm{ds} \right| \\ &\le \int_0^1 \sum_{i=1}^n |\vartheta(t_i,s)-\vartheta(t_{i-1},s)||x(s)|\mathrm{ds},\end{aligned}$$

where $\vartheta \colon I \times I \to \mathbb{R}$ is given by the following formula

$$\vartheta(t,s) = \begin{cases} k(t,s), & \text{if } (t,s) \in \Delta, \\ 0, & \text{if } (t,s) \notin \Delta. \end{cases} \tag{8}$$

Since

$$\sum_{i=1}^n |\vartheta(t_i,s)-\vartheta(t_{i-1},s)| \le |k(s,s)| + \bigvee_s^1 k(\cdot,s) \le m_2(s) \quad \text{for a.e. } s \in I,$$

we infer that

$$\sum_{i=1}^n |Kx(t_i)-Kx(t_{i-1})| \le \int_0^1 m_2(s)|x(s)|\mathrm{ds}.$$

This proves that $\|Kx\|_{BV_1} \le \|m_2\|_{L^q} \cdot \|x\|_{L^p}$, which means that K maps the space $L^p(I)$ into $BV_1(I)$ and is continuous. □

Lemma 4 *Let $p \in (1, +\infty]$. Suppose that the assumptions 5° (a) and (b) hold. If a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $L^p(I)$ converges almost everywhere (or in measure) to a function $x \in L^p(I)$, then the sequence $(Kx_n)_{n\in\mathbb{N}}$, where K is given by (7), converges to Kx with respect to the BV_1-norm.*

Proof First, let us note that $Kx, Kx_n \in BV_1(I)$ by Lemma 3. If $0 = t_0 < \cdots < t_n = 1$ is an arbitrary finite partition of the interval I, then

$$\sum_{i=1}^{n}|Kx(t_i) - Kx_n(t_i) - Kx(t_{i-1}) + Kx_n(t_{i-1})|$$
$$\leq \int_0^1 \sum_{i=1}^{n}\big|\vartheta(t_i, s) - \vartheta(t_{i-1}, s)\big|\big|x_n(s) - x(s)\big|\mathrm{ds},$$

where the function $\vartheta\colon I \times I \to \mathbb{R}$ is defined by (8). Hence

$$\bigvee_0^1[Kx_n - Kx] \leq \int_0^1 m_2(s)\big|x_n(s) - x(s)\big|\mathrm{ds}$$

(cf. the proof of Lemma 3), which, in view of the assumptions and Vitali's convergence theorem (see [20, Theorem 6.2.12]), shows that $\|Kx_n - Kx\|_{BV_1} \to 0$ as $n \to +\infty$. □

The following example shows that Lemma 4 is false if $p = 1$.

Example 2 Let us consider a sequence $(x_n)_{n\in\mathbb{N}}$ of Lebesgue integrable functions defined by the formulas

$$x_n(t) = n \cdot \chi_{[0,1/n]}(t) \qquad \text{for} \quad t \in I \quad \text{and} \quad n \in \mathbb{N}.$$

Clearly, the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in $L^1(I)$ and converges almost everywhere to the zero function. If we take the kernel k equal identically to 1 on Δ, then

$$Kx_n(t) = \int_0^t x_n(s)\mathrm{ds} = n \cdot \min(t, 1/n) \qquad \text{for} \quad t \in I,$$

which shows that $\bigvee_0^1 Kx_n = 1$ for $n \in \mathbb{N}$. Therefore, the sequence $(Kx_n)_{n\in\mathbb{N}}$ cannot converge to the zero function with respect to the BV_1-norm.

Corollary 2 *If the kernel $k\colon \Delta \to \mathbb{R}$ satisfies the assumptions 5° (a) and (b) with $q = 1$, then the integral operator $K\colon BV_1(I) \to BV_1(I)$ given by (7) is compact.*

In the proof of the main result of this subsection, we will need a Vidossich-type result. Assume that D is a bounded and convex subset of a normed space, and E is a Banach space. Denote by $C(D, E)$ the space of all bounded and continuous functions $x\colon D \to E$, endowed with the supremum norm, similarly as in the case of real-valued functions.

Theorem 7 ([28, Theorem 2]) *Let $F\colon C(D, E) \to C(D, E)$ be a continuous mapping satisfying the following conditions:*

(i) *the set $F(C(D, E))$ is equiuniformly continuous;*
(ii) *there exist $t_0 \in D$ and $x_0 \in E$ such that $F(x)(t_0) = x_0$ for every $x \in C(D, E)$;*
(iii) *for every $\varepsilon > 0$ and $x, y \in C(D, E)$ the following implication holds*

$$x|_{D_\varepsilon} = y|_{D_\varepsilon} \quad \Rightarrow \quad F(x)|_{D_\varepsilon} = F(y)|_{D_\varepsilon},$$

where $D_\varepsilon = \{t \in D : \|t - t_0\| \leq \varepsilon\}$;

(iv) *every sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $C(D, E)$ *such that* $\lim_{n\to\infty}(x_n - F(x_n)) = 0$ *has a limit point.*

Then the set of fixed points of the mapping F is a compact R_δ, that is, it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.

Let us refer the reader interested in other important results, similar to Theorem 7, to the papers [6] and [29].

Theorem 8 *If the assumptions* $3°$–$5°$ *hold, then the set T of all continuous solutions of bounded variation in the sense of Jordan to the nonlinear Volterra–Hammerstein integral equation (6) is a compact R_δ in the Banach space $C(I)\cap BV_1(I)$ endowed with the BV_1-norm.*

Proof The proof falls into two parts. First, we shall show that the mapping

$$F(x)(t) = g(t) + \int_0^t k(t, s) f(s, x(s))\mathrm{ds}, \qquad t \in I,$$

defined for $x \in C(I)$ satisfies the assumptions of Theorem 7; let us note that the mapping F is well defined, that is, the above integral exists and is finite for every $x \in C(I)$ and $t \in I$ (cf. Remark 8).

Let $x \in C(I)$. Given $\varepsilon > 0$, in view of the assumptions, there exists $\delta > 0$ such that:

- $|g(t) - g(\tau)| \le \frac{1}{3}\varepsilon$ for $t, \tau \in I$ such that $|t - \tau| \le \delta$;
- $\int_0^t |k(\tau, s) - k(t, s)| m_1(s)\mathrm{ds} \le \frac{1}{3}\varepsilon$ for $(\tau, t) \in \Delta$ such that $|t - \tau| \le \delta$;
- $\int_A m_1(s)m_2(s)\mathrm{ds} \le \frac{1}{3}\varepsilon$ for any Lebesgue measurable set $A \subseteq I$ such that $\mu(A) \le \delta$.

Therefore, if $t, \tau \in I$ are such that $0 \le \tau - t \le \delta$, then

$$\begin{aligned}
|F(x)(t) - F(x)(\tau)| &\le |g(t) - g(\tau)| + \left| \int_0^t k(t, s) f(s, x(s))\mathrm{ds} - \int_0^\tau k(\tau, s) f(s, x(s))\mathrm{ds} \right| \\
&\le |g(t) - g(\tau)| + \int_0^t |k(t, s) - k(\tau, s)||f(s, x(s))|\mathrm{ds} \\
&\quad + \int_t^\tau |k(\tau, s)||f(s, x(s))|\mathrm{ds} \\
&\le |g(t) - g(\tau)| + \int_0^t |k(t, s) - k(\tau, s)| m_1(s)\mathrm{ds} + \int_t^\tau m_1(s)m_2(s)\mathrm{ds} \\
&\le \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon.
\end{aligned}$$

This shows that $F(x) \in C(I)$. Furthermore, let us observe that the number δ in the above reasoning is independent of x, which implies that the set $F(C(I))$ is equiuniformly continuous.

The continuity of the mapping F is a consequence of Lemma 4 and the fact that if a sequence $(x_n)_{n\in\mathbb{N}}$ in $C(I)$ is uniformly convergent to $x \in C(I)$, then the sequence $\big(f(\cdot, x_n(\cdot))\big)_{n\in\mathbb{N}}$, which is bounded in $L^p(I)$, converges a.e. to the function $t \mapsto f(t, x(t))$, $t \in I$.

The assumptions (ii) and (iii) of Theorem 7 are obviously satisfied, if we set $t_0 = 0$ and $x_0 = g(0)$.

Hence, it suffices to prove that the mapping F satisfies the Palais–Smale condition. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $C(I)$ such that $\lim_{n\to\infty}(x_n - F(x_n)) = 0$ with respect to the supremum norm. In view of the assumption $4°$ and Lemma 3, we get

$$\|F(x)\|_{BV_1} \le \|g\|_{BV_1} + \|m_1\|_{L^p} \cdot \|m_2\|_{L^q} \quad \text{for} \quad x \in C(I). \tag{9}$$

Therefore, by Helly's selection theorem there exists a subsequence $(F(x_{n_k}))_{k\in\mathbb{N}}$ of $(F(x_n))_{n\in\mathbb{N}}$ pointwise convergent to a function $y \in BV_1(I)$. Thus, $(x_{n_k})_{k\in\mathbb{N}}$ is also pointwise convergent to y. Hence, for a.e. $t \in I$, we have $f(t, x_{n_k}(t)) \to f(t, y(t))$ as $k \to +\infty$, and the sequence $(f(\cdot, x_{n_k}(\cdot)))_{k\in\mathbb{N}}$ is bounded in $L^p(I)$. This, by Lemma 4, implies that $(F(x_{n_k}))_{k\in\mathbb{N}}$ converges to $F(y) = y$ with respect to the BV_1-norm, and so, since the supremum norm is weaker than the BV_1-norm, the sequence $(x_n)_{n\in\mathbb{N}}$ has a limit point in $C(I)$.

All the assumptions of Theorem 7 are satisfied, and therefore, the set S of all continuous solutions to Eq. (6) is a compact R_δ in $C(I)$. To end the proof, it suffices to show that S endowed with the metric d_∞ induced by the supremum norm is homeomorphic to the set T of all continuous solutions to (6) of bounded Jordan variation, endowed with the metric d_{BV_1} induced by the BV_1-norm. Note that $S = T$ as sets (cf. the formula (9)), and since the BV_1-norm is stronger than the supremum norm, we get that the identity map $\mathrm{id}\colon T \to S$ is continuous. Now, we shall show that $\mathrm{id}\colon S \to T$ is also continuous. Let us take a sequence $(x_n)_{n\in\mathbb{N}}$ in S convergent to $x_0 \in S$. Reasoning as above, we infer that the sequence $(F(x_n))_{n\in\mathbb{N}}$ converges to $F(x_0)$ with respect to d_{BV_1}. But $F(x_n) = x_n$ for all $n \in \mathbb{N} \cup \{0\}$, and hence $\|x_n - x_0\|_{BV_1} \to 0$ as $n \to +\infty$. This shows that the identity map constitutes a homeomorphism between the metric spaces T and S and, in consequence, it proves that the set of all continuous solutions to (6) of bounded variation in the sense of Jordan is a compact R_δ set in $C(I) \cap BV_1(I)$ with respect to the BV_1-norm. □

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